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# Cubic Spline Interpolation of Continuous Functions

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Let [0, 1] be partitioned into subintervals $h_1, ..., h_n$. Let $P_n$ be an associated cubic spline interpolation operator defined on the space $C[0, 1]$. Let $h_0 = h_n$ and $m_n = \max\{h_i/h_j : |i - j| = 1\}$. Examples are given for which $m_n$ is uniformly bounded as $n$ tends to infinity while $\| P_n \|$ is unbounded. The periodic cubic spline interpolation operator is shown to have uniformly bounded norm if $m_n \leqslant 2.439$ for all $n$.

## 1. INTRODUCTION

Let $f$ be continuous on [0, 1] and $\pi_n$: $0 = x_0 < x_1 < \cdots < x_n = 1$ be a partitioning of [0, 1]. A function $s$ is a *cubic spline interpolant* associated with $f$ and $\pi_n$ if

(a) $s \in C^2[0, 1]$;

(b) $s(x)$ is a cubic polynomial on $(x_{i-1}, x_i)$ for $i = 1, ..., n$; and

(c) $s(x_i) = f(x_i)$ for $i = 0, 1, ..., n$.

The two free parameters in a cubic spline interpolant can be variously assigned. Three common ways follow.

DEFINITION 1. Let $s = N_n f$ be the cubic spline interpolant to $f$ prescribed by (a), (b), (c) and

($d_1$) $s''(0) = s''(1) = 0$.

DEFINITION 2. Let $s = S_n f$ be the cubic spline interpolant to $f$ prescribed by (a), (b), (c) and

($d_2$) $s'(0) = s'(1) = 0$.

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DEFINITION 3. Let $s = L_n f$ be the cubic spline interpolant to $f$ prescribed by (a), (b), (c) and

$$(d_3) \quad s'(0) = s'(1) \text{ and } s''(0) = s''(1).$$

As $f$ ranges over $C[0, 1]$, these definitions specify $N_n$, $S_n$, $L_n$ as linear idempotent operators from $C[0, 1]$ onto the corresponding cubic spline subspaces of dimension $n + 1$. The subspace defined by (a), (b), ($d_1$) consists of the natural cubic splines (under the supremum norm). If one restricts $C[0, 1]$ to the continuous functions satisfying $f(0) = f(1)$, then $L_n$ becomes the periodic cubic spline operator and the spline subspace has dimension $n$.

One concern in the area of cubic spline interpolation is: As $n \to \infty$ and $\pi_n = \max_i (x_i - x_{i-1}) \to 0$, what conditions on a sequence $\{\pi_n\}$ of partitions will guarantee that $\limsup \| L_n \| < \infty$ or, equivalently (see [5]), that $\lim \| L_n f - f \| = 0$ for $f \in C[0, 1]$?

Let $h_i = x_i - x_{i-1}$ for $i = 1,..., n$ and $h_0 = h_n$. Let

$$K_n = \max\{h_i/h_j : i, j = 1,..., n\}$$

and

$$m_n = \max\{h_i/h_j : | i - j | = 1 \text{ and } i, j = 0,..., n\}.$$

Sharma and Meir [11] have shown that

$$K_n \leqslant K < \infty \tag{1}$$

is a sufficient condition that

$$\limsup \| L_n \| < \infty \quad \text{or} \quad \lim \| L_n f - f \| = 0 \quad \text{for} \quad f \in C[0, 1]. \tag{2}$$

Nord [8] has shown that there exists a sequence $\{\pi_n\}$ for which both (1) and (2) do not hold.

It was demonstrated by Cheney and Schurer [4, Test Case 3] that (2) could hold while (1) was invalid. Then, in succession, it was shown that

$m_n \leqslant m < \sqrt{2}$ (Meir and Sharma [7]);

$m_n \leqslant m < 2$ (Cheney and Schurer [5]); and

$m_n \leqslant m < 1 + \sqrt{2}$ (Hall [6])

are sufficient conditions that (2) hold.

Conditions which would imply the Cheney–Schurer result had been stated by Birkhoff and de Boor [2, corollary following Theorem 1].

In Section 2 below we prove that $m_n \leqslant m < \infty$ is *not* a sufficient condition for (2) to hold.

THEOREM 1. *For each fixed* $m > (3 + \sqrt{5})/2$ *there exists a sequence* $\{\pi_n\}$ *for which* $m_n \leqslant m$ *for all* $n$ *while*

$$\limsup \| N_n \| = \limsup \| S_n \| = \limsup \| L_n \| = \infty.$$

In Section 3 we use $B$-splines to establish the following theorem.

THEOREM 2. *If* $m < 2.439+$ *and* $m_n \leqslant m$ *for all* $n$, *then*

$$\| L_n \| \leqslant \frac{2(1 + m)(2 + m)(1 + m + m^2)}{6m + 7m^2 + m^3 - 2m^4}.$$

The approach does not apply to the operators $N_n$ or $S_n$.

## 2. PROOF OF THEOREM 1

To prove Theorem 1 we use Test Case 4 in [4] with $\theta^{-1} > (3 + \sqrt{5})/2$ and place a lower bound on $\| L_n \|$ (respectively, $\| N_n \|$, $\| S_n \|$) which is of the form $\alpha^n$ with $\alpha > 1$.

Let $P_n$ denote one of the operators $N_n$, $L_n$, $S_n$, and let $s_0, s_1, \ldots, s_n$ be the interpolating basis for the corresponding subspace. (If our concern is with periodic splines, we ignore $s_0$ here and henceforth.) Then

$$s_i(x_j) = \delta_{ij} \qquad \text{for} \quad i, j = 0, 1, \ldots, n \tag{3}$$

and

$$\| P_n \| = \max \sum | s_i(x)| \geqslant | s_0(\tfrac{1}{2}) + s_n(\tfrac{1}{2})|.$$

This inequality is the first step in our proof.

Let $m \geqslant 1$ and let $n = 2k + 1$ be an odd integer. Let

$$h_{1=1}/(2 + 2m + \cdots + 2m^{k-1} + m^k)$$

and $h_{i+1} = h_{n-i} = m^i h_1$ for $i = 0, 1, \ldots, k$. Let $\pi_n$ be defined by setting $x_i = h_1 + \cdots + h_i$ for $i = 0, \ldots, n$.

Set $s = s_0 + s_n$ and $\mu_i = s'(x_i)$ for $i = 0, \ldots, n$. On $(x_k, x_{k+1})$ we have

$$s(x) = (x_{k+1} - x)(x - x_k)[\mu_k(x_{k+1} - x) - \mu_{k+1}(x - x_k)]/h_{k+1}^2.$$

From symmetry, $s(x) = s(1 - x)$. Hence,

$$s(\tfrac{1}{2}) = \mu_k h_{k+1}/4 = m^k \mu_k h_1/4. \tag{4}$$

Thus, we can place a lower bound on $\| P_n \|$ by finding $\mu_k$.

LEMMA 1. *Let* $\beta = (m+1) + (m^2 + m + 1)^{1/2}$. *If* $P_n = N_n$, *then*

$$\mu_k = 3(-m\beta)^k(\beta^2 - m)/(h_1 D_1), \tag{5}$$

*where*

$$D_1 = (\beta + m)\,\beta^{2k+1} - (\beta + 1)\,m^{k+1}.$$

*If* $P_n = L_n$ *or* $S_n$, *then*

$$\mu_k = 3(-m\beta)^k(\beta^2 - m)/(h_1 D_2), \tag{6}$$

*where*

$$D_2 = (\beta - 1)\,\beta^{2k+1} + (\beta - m)\,m^k.$$

*Proof.* From (a), (b), (c) and (3) we have the relations (see [1, p. 12])

$$m\mu_0 + 2(1+m)\,\mu_1 + \mu_2 = -3m/h_1 \tag{7a}$$

and

$$m\mu_{i-1} + 2(1+m)\,\mu_i + \mu_{i+1} = 0 \qquad \text{for} \quad i = 2,\dots, k. \tag{7b}$$

A solution of (7b) is

$$\mu_i = -\mu_{n-i} = A(-\beta)^i + B(-m/\beta)^i \qquad \text{for} \quad i = 1,\dots, k+1, \tag{8}$$

where $A$ and $B$ are arbitrary constants and $\beta$ is the larger solution of

$$x^2 - 2(m+1)x + m = 0. \tag{9}$$

From (8) with $i = k, k+1$ we have

$$\mu_k = A(-\beta)^k + B(-m/\beta)^k = -A(-\beta)^{k+1} - B(-m/\beta)^{k+1}$$

or

$$(\beta - 1)\,\beta^{2k+1}A - (\beta - m)\,m^k B = 0. \tag{10}$$

From (7a) and (9) we have

$$A + B - \mu_0 = 3/h_1 . \tag{11}$$

If $P_n = N_n$, Definition 1 requires that $s''(0) = 0$, yielding

$$\mu_1 + 2\mu_0 = -3/h_1$$

or

$$\beta^2 A + mB - 2\beta\mu_0 = 3\beta/h_1 . \tag{12}$$

Solving (11) for $\mu_0$ and substituting into (12) yields in conjunction with (10) that

$$A = 3(\beta - m)m^k/(h_1 D_1)$$

and

$$B = 3(\beta - 1)\beta^{2k+1}/(h_1 D_1).$$

Substitution into (8) gives (5).

The proof of (6) is similar with $\mu_0 = 0$ required.

LEMMA 2. *If $P_n = N_n$, $S_n$, or $L_n$, then*

$$| s(\tfrac{1}{2})| = (-m)^k u_k h_1/4 > (\tfrac{3}{8})(m^2/\beta)^k.$$

*Proof.* Suppose first that $P_n = N_n$. Then, from (4) and (5)

$$| s(\tfrac{1}{2})| = (-m)^k u_k h_1/4 = 3m^{2k}\beta^k(\beta^2 - m)/(4D_1).$$

Dropping the term $(\beta + 1)\, m^{k+1}$ from $D_1$ yields

$$| s(\tfrac{1}{2})| > (\tfrac{3}{4})(\beta^2 - m)(m^2/\beta)^k/(\beta^2 + m\beta).$$

Since $(\beta^2 - m)/(\beta^2 + m\beta) > \frac{1}{2}$, the result follows.

Similarly, if $P_n = S_n$ or $L_n$, we replace the term $(\beta - m)\, m^k$ in $D_2$ by the larger term $(\beta - m)\, \beta^{2k}$ to get

$$| s(\tfrac{1}{2})| > (\tfrac{3}{4})(m^2/\beta)^k.$$

Since $m^2/\beta > 1$ and $m^2 - 3m + 1 > 0$ are equivalent statements, Lemma 2 immediately implies Theorem 1.

The above construction does not satisfy the requirement that $| \pi_n | \to 0$. However, adjoining $k$ copies of $\pi_n$ produce a partitioning of $[0, k]$ which can be contracted into a new partitioning of $[0, 1]$ which does satisfy this requirement for $n = k(2k + 1)$.

There are many sequences $\{\pi_n\}$ for which a comparable theorem is not true. Indeed, Hall [6] has constructed a sequence for which (2) holds although $\lim K_n = \infty$ and $m_n = 3$ for all $n$.

## 3. PROOF OF THEOREM 2

The question of sufficiency for $m$ between $1 + \sqrt{2} = 2.41+$ and $(3 + \sqrt{5})/2 = 2.62-$ is still open. We shall use the *normalized B-spline basis* (see [9]) to narrow this range.

The normalized $B$-splines $\sigma_1, \ldots, \sigma_n$ are defined by

$$\sigma_i = a_{i,i-1}s_{i-1} + a_{ii}s_i + a_{i,i+1}s_{i+1} \qquad \text{for} \quad i = 1, \ldots, n$$

where

$$a_{i,i-1} = \frac{h_{i-1}^2}{(h_{i-1} + h_i)(h_{i-1} + h_i + h_{i+1})},$$

$$a_{i,i+1} = \frac{h_{i+2}^2}{(h_{i+1} + h_{i+2})(h_i + h_{i+1} + h_{i+2})},$$

and

$$a_{ii} = 1 - a_{i-1,i} - a_{i+1,i}.$$

Here and henceforth, subscripts are to be read modulo $n$. In particular,

$$\sigma_1 = a_{1n}s_n + a_{11}s_1 + a_{12}s_2$$

and

$$\sigma_n = a_{n,n-1}s_{n-1} + a_{nn}s_n + a_{n1}s_1.$$

Let $A$ denote the matrix $(a_{ij})$ with zeros in the unspecified entries and denote its inverse by $A^{-1} = (b_{ij})$. Then we have the inverse representation

$$s_i = \sum_j b_{ij}\sigma_j \qquad \text{for} \quad i = 1, \ldots, n.$$

If we set $x_+ = (x + |x|)/2$ and

$$\omega_i(x) = (x - x_{i-2}) \cdots (x - x_{i+2}),$$

the $\sigma_i$ are given on $[x_{i+2} - 1, x_{i-2} + 1]$ by

$$\sigma_i(x) = (x_{i+2} - x_{i-2}) \sum_{j=i-2}^{i+2} \frac{(x_j - x)_+^3}{\omega_i'(x_j)}$$

with $\sigma_i(x) = \sigma_i(x + 1)$ for all real $x$. These functions have the property that

$$\sum |\sigma_i(x)| = \sum \sigma_i(x) = 1 \qquad \text{for all } x.$$

Since

$$\begin{aligned} \sum_i |s_i(x)| &= \sum_i \Big| \sum_j b_{ij}\sigma_j(x) \Big| \\ &\leqslant \sum_j \Big( \sum_i |b_{ij}| \Big) \sigma_j(x) \\ &\leqslant \max_j \sum_i |b_{ij}| \\ &= \| A^{-1} \|_1, \end{aligned}$$

we have

LEMMA 3. $\| L_n \| \leqslant \| A^{-1} \|_1$.

Thus, a bound on $\| A^{-1} \|_1$ suffices as a bound on $\| L_n \|$. To prove Theorem 2 we choose

$$D = \operatorname{diag}\{1/a_{ii}\}$$

and use the bound

$$\| A^{-1} \| \leqslant \| D \| / (1 - \| I - DA \|). \tag{13}$$

Here and henceforth, all matrix norms are columns norms. Since $A$ is the transpose of an oscillation matrix, more efficient bounds on $\| A^{-1} \|$ may exist.

To use (13) we must show that

$$\| I - DA \| < 1 \text{ for } m_n \text{ sufficiently small.} \tag{14}$$

Assuming without loss of generality that $\min a_{ii} = a_{22}$, we have

$$\begin{aligned} 1/\| D \| &= a_{22} = 1 - a_{12} - a_{32} \\ &= 1 - \frac{h_3^2}{(h_2 + h_3)(h_1 + h_2 + h_3)} - \frac{h_2^2}{(h_2 + h_3)(h_2 + h_3 + h_4)} \\ &\geqslant 1 - \frac{m^3}{(1 + m)(1 + m + m^2)} - \frac{1}{(1 + m)(2 + m)} \\ &= \frac{(2m + 1)(m + 1)}{(m^2 + m + 1)(m + 2)} \end{aligned}$$

or

$$\| D \| \leqslant \frac{(m^2 + m + 1)(m + 2)}{(2m + 1)(m + 1)}.$$

Here we have repeatedly used the restrictions

$$1/m \leqslant h_i/h_{i-1} \leqslant m,$$

observing that the choice

$$h_3 = mh_2 = mh_4 = m^2 h_1$$

minimizes $a_{22}$.

Assuming, again without loss of generality, that $I - DA$ attains its norm in the second column gives

$$\begin{aligned} \| I - DA \| &= a_{12}/a_{11} + a_{32}/a_{33} \\ &\leqslant \frac{2m^4 + 3m^3 + 3m^2 + 2m + 2}{2(2m + 1)(m + 1)^2} \end{aligned}$$

by a procedure similar to that indicated above. Thus,

$$1 - \| I - DA \| = \frac{-2m^4 + m^3 + 7m^2 + 6m}{2(2m + 1)(m + 1)^2}.$$

Combining the results of this section yields Theorem 2.

## 4. Remarks

We close with two remarks about quintic spline interpolation.

To get an analog of Theorem 1 for quintic spline interpolation, it is convenient to use Eqs. (9) and (10) of Schurer [10]. Preliminary efforts in this direction suggest that the quantity $m^2/\beta$ in Lemma 2 will be replaced by $m^3/\gamma$ where $\gamma$ is a root of a fourth-degree polynomial analogous to (9) above and that the quantity $(3 + \sqrt{5})/2 = 2.62-$ of Theorem 1 will be replaced by $5.60+$. The latter number is a root of an eight-degree reciprocal polynomial.

Concerning an analog of Theorem 2, one notes that if the matrix $A$ is suitably reinterpreted, Lemma 3 is valid for periodic quintic splines as well. See Richards [9] for a description of the normalized $B$-spline basis in this case. Since $A$ is a cyclic-variation-diminishing matrix, its minors of odd order have positive determinant (see [9]). Thus, one may use a Lemma of de Boor's [3, p. 457] to bound $\| A^{-1} \|$. The advantage is as follows: For the choice

$$D = \operatorname{diag}\{1/a_{ii}\},$$

(13) and (14) above would require that (for example)

$$a_{13}/a_{11} + a_{23}/a_{22} + a_{43}/a_{44} + a_{53}/a_{55} < 1;$$

whereas, the corresponding use of de Boor's Lemma would result in the relaxed restriction

$$-a_{13}/a_{11} + a_{23}/a_{22} + a_{43}/a_{44} - a_{53}/a_{55} < 1.$$

In developing analogs of Theorem 2, one should also consider the method used by Hall in [6].

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